RELATIONS BETWEEN SOME CONSTANTS ASSOCIATED WITH FINITE DIMENSIONAL BANACH SPACES

BY

D. J. H. GARLING AND Y. GORDON*

ABSTRACT

We define the asymmetry constant $s(E)$ of a Banach space E, and show examples of finite-dimensional spaces with "large" asymmetry constants. If E is n-dimensional, $\lambda(E)$ its projection constant and $\pi_1(I_E)$ the absolutely summing norm of the identity operator I_E, then $n \leqq \lambda(E)\pi_1(I_E) \leq n(s(E))^2$. Similar equations linking the p-absolutely summing and the nuclear norms of I_E are established. We also obtain estimates on these norms, for example $\pi_2(I_E) = \sqrt{n}$.

1. Preliminaries and definitions

The results obtained here are related to those in [4] and [5]. We recall some basic definitions and results. We denote the Banach space of all continuous linear operators from a Banach space E into a Banach space F by $L(E,F)$ and denote the identity operator on a Banach space E by I_E. If $T \in L(E,F)$, T is said to be *absolutely summing* [15] if there exists a non-negative number C such that

$$\sum_{i=1}^{k} \| Tx_i \| \leqq C \sup_{\|a\| \leqq 1} \sum_{i=1}^{k} |\langle x_i, a\rangle|,$$

for each finite sequence $x_1, \cdots, x_k$ in E. If $\pi_1(T)$ denotes the smallest possible value of C, then the space $\pi_1(E,F)$ of all absolutely summing operators from E into F is a Banach space under the norm π_1. A linear operator is absolutely summing if and only if it is quasi-integral in the sense of [14], and $\pi_1(T) = i_1^Q(T)$. I_E is absolutely summing if and only if E is finite-dimensional [1]; if E is finite dimensional, let $\pi_1(E) = \pi_1(I_E)$.

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T is said to have the *extension property* if whenever E is isometrically embedded in a Banach space E_1, T can be extended to a continuous linear operator T_1 from E_1 into F. E can always be isometrically embedded in a space $C(H)$, where H is compact, Hausdorff and extremally disconnected. T has the extension property if (and only if) it can be extended to a continuous linear operator from $C(H)$ into F. If $c(T)$ denotes the infimum of the norm of all such extensions, the space $L_c(E, F)$ of all operators from E to F with the extension property is a Banach space under the norm c. $c(I_E) = \lambda(E)$, the *projection constant* of E. For details, see [12]. If F is reflexive (and in particular if F is finite-dimensional) it is sufficient to suppose that the space H above is compact and Hausdorff. Thus if F is reflexive, T has the extension property if and only if T is ∞-*integral* in the sense of [14], with equality of norms.

It therefore follows from the duality results of [14] that if E is a real n-dimensional space, the dual of $(L(E, E), \pi_1)$ may be identified with $(L(E, E), c)$, the pairing being given by

$$\langle S, T \rangle = Tr(ST).$$

Thus,

(1) $\pi_1(T) = \sup\{ |Tr(ST)| : c(S) \leqq 1\}$ and

(2) $c(T) = \sup\{ |Tr(ST)| : \pi_1(S) \leqq 1\}$.

In particular,

(3) $\pi_1(E) = \sup\{ |Tr(T)| : T \in L(E, E),\ c(T) \leqq 1\}$ and

(4) $\lambda(E) = \sup\{ |Tr(T)| : T \in L(E, E),\ \pi_1(T) \leqq 1\}$.

From either of these equations it follows that

(5) $\lambda(E)\pi_1(E) \geqq n$ ([4] Theorem 2).

We shall see that in certain circumstances this inequality can be replaced by equality.

We recall that the *distance coefficient* $d(X, Y)$ of two Banach spaces X and Y is defined to be $\inf(\| T \| \, \| T^{-1} \|)$, the infimum being taken over all linear homeomorphisms T of X onto Y.

We now suppose that E is a finite-dimensional real Banach space. If $x_1, x_2, \cdots, x_k$ is a finite sequence in E,

$$\sup\left\{ \sum_{i=1}^{k} |\langle x_i, a \rangle| \, ; \, \| a \| \leqq 1 \right\} = \sup\left\{ \left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| ; \, \varepsilon_i = \pm 1 \right\},$$

so that

$$\pi_1(E) = \sup\left\{ \sum_{i=1}^{k} \| x_i \| \Big/ \sup_{\varepsilon_i = \pm 1} \| \Sigma \varepsilon_i x_i \| \right\},$$

the supremum being taken over all finite sequences in E. Note that $(\pi_1(E))^{-1}$ is equal to the constant $\mu_1(E)$ introduced in [5].

We recall that the *Macphail* number $\mu(E)$ is defined as

$$\mu(E) = \inf\left\{\sup_J \left\| \sum_{i\in J} x_i \right\| \Big/ \sum_{i=1}^{k} \|x_i\|\right\}$$

the supremum being taken over all subsets J of $\{1,\cdots,k\}$ and the infimum over all finite sequences $x_1,\cdots,x_k$ $(k = 1,2,\cdots)$ in E ([4], [16]).

PROPOSITION. $\pi_1(E) = (2\mu(E))^{-1}$.

This is a direct consequence of the following easily established identities:

Let $x_1,\cdots,x_n$ be n vectors in a real normed space E. Let $y_1,\cdots,y_{2n}$ be the vectors $x_1,\cdots,x_n, -x_1,\cdots,-x_n$. Then

$$\sup_J \left\| \sum_{i\in J} y_i \right\| = \sup_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| = \tfrac{1}{2} \sup_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^{2n} \varepsilon_i y_i \right\|, \tag{6}$$

$$\pi_1(E) = \sup\left\{\sum_{i=1}^{2n} \|y_i\| \Big/ \sup_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^{2n} \varepsilon_i y_i \right\|\right\}, \tag{7}$$

$$(\mu(E))^{-1} = \sup\left\{\sum_{i=1}^{2n} \|y_i\| \Big/ \sup_J \left\| \sum_{i\in J} y_i \right\|\right\}, \tag{8}$$

the suprema in (7), (8) being taken over all sequences $(y_1,\cdots,y_{2n}) = (x_1,\cdots,x_n, -x_1,\cdots,-x_n)$.

Finally, if G is a subset of $L(E,E)$, we denote by G' the set of elements of $L(E,E)$ which commute with every element of G.

2. Spaces with enough symmetries

If E is a Banach space, let G_E denote the group of linear isometries of E onto itself. We shall say that E has enough symmetries if $G'_E = \{\lambda I_E\}$.

THEOREM 1. *If E is an n-dimensional real normed space with enough symmetries, $\lambda(E) = 2n\mu(E)$.*

PROOF. G_E is a compact group: let ν be the normalized Haar measure on G_E. There exists $T \in L(E,E)$ such that $\pi_1(T) = 1$ and $\lambda(E) = Tr(T)$. If $g \in G_E$, $\pi_1(g^{-1}Tg) = 1$, and $Tr(g^{-1}Tg) = Tr(T) = \lambda(E)$. Let $T_0 = \int_{G_E}(g^{-1}Tg)d\nu(g)$. Then $\pi_1(T_0) \leqq 1$, and $Tr(T_0) = \lambda(E)$. Since T_0 commutes with every element of G_E, T_0 is a scalar multiple of the identity; clearly, $T_0 = n^{-1}\lambda(E)I_E$, and so

$$1 \geqq \pi_1(T_0) = n^{-1}\lambda(E)\pi_1(I_E) = (2n\mu(E))^{-1}\lambda(E),$$

from which the result follows.

Let us apply this result to the finite dimensional l^p spaces. It is easy to calculate $\mu(l_n^1)$ and $\mu(l_n^2)$:

$$\mu(l_n^2) = \frac{\Gamma(\frac{1}{2}n)}{2\sqrt{\pi}\,\Gamma(\frac{1}{2}n+\frac{1}{2})} = \frac{1}{2\pi}\int_0^\pi \sin^{n-1}\theta d\theta = \frac{\alpha_2(n)}{\sqrt{n}}$$

where $\alpha_2(n)$ decreases monotonically to the limit $1/\sqrt{(2\pi)}$ ([3], [5], and [16]) and

$\mu(l_n^1) = \mu(l_n^2)$ if n is odd,

$= \mu(l_{n+1}^2)$ if n is even [5].

Applying Theorem 1, we see that

$$\lambda(l_n^2) = \frac{n\Gamma(\frac{1}{2}n)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n+\frac{1}{2})} = 2\sqrt{n}\,\alpha_2(n) \sim \sqrt{\frac{2n}{\pi}}$$

and that $\lambda(l_n^1) = \lambda(l_n^2)$ if n is odd,

$= \lambda(l_{n+1}^2)$ if n is even.

The first of these results is due to Grünbaum [7] and Rutovitz [16]; the second to Grünbaum [7]; their proofs are quite different.

More trivially, since $\lambda(l_n^\infty) = 1$ it follows that $\mu(l_n^\infty) = 1/2n$. Rutovitz ([16] Theorem 1) showed that if $p \geqq 2$, $\mu(l_n^p) = n^{-(1-1/p)}\alpha_p(n)$, where $1/\sqrt{(2\pi)} < \alpha_2(n) \leqq \alpha_p(n) \leqq \alpha_\infty(n) = \frac{1}{2}$ and $\alpha_2(n)$ decreases to $1/\sqrt{(2\pi)}$ as $n \to \infty$. From Theorem 1 we conclude that $\lambda(l_n^p) = 2n^{1/p}\alpha_p(n)$: this strengthens [16] Theorem 3 slightly. We shall return to these spaces in §5.

3. The asymmetry of a Banach space

If a Banach space E does not have enough symmetries, it is possible to give a useful measure of its lack of symmetry. We define the *asymmetry constant* $s(E)$ to be the infimum of all those numbers μ with the following property: there is a group G of invertible operators on E such that $G' = \{\lambda I_E\}$ and $\sup_{g \in G}\|g\| \leqq \mu$. (If there is no such μ, we set $s(E) = \infty$). We can characterize $s(E)$ in the following way.

THEOREM 2. *Let* $\sigma(E) = \inf\{d(E,F): F$ *a Banach space with enough symmetries*$\}$. *Then* $\sigma(E) = s(E)$.

(If E is not linearly homeomorphic to a Banach space with enough symmetries, we set $\sigma(E) = \infty$.)

If $\sigma(E)$ is finite and $\varepsilon > 0$, there is a linear homeomorphism T of E onto a Banach space F with enough symmetries such that $\|T\| \, \|T^{-1}\| \leqq \sigma(E) + \varepsilon$. Let $G = \{T^{-1}hT : h \in G_F\}$. Then G is a group of invertible operators on E, $G' = \{\lambda I_E\}$, and $\sup_{g \in G} \|g\| \leqq \sigma(E) + \varepsilon$. Thus, $s(E) \leqq \sigma(E)$.

Conversely, suppose that G is any group of invertible operators on E such that $G' = \{\lambda I_E\}$ and $\sup_{g \in G} \|g\| \leqq \mu$. For each $x \in E$, let $\|x\|_1 = \sup\{\|gx\| : g \in G\}$. $\|x\|_1$ is a norm on E, and $\|x\| \leqq \|x\|_1 \leqq \mu \|x\|$. Thus, if E_1 denotes the space $(E, \| \, \|_1)$, $d(E, E_1) \leqq \mu$. Now, if $g_0 \in G$ and $x \in E$,

$$\|g_0 x\|_1 = \sup_{g \in G} \|g g_0 x\| = \sup_{g \in G} \|gx\| = \|x\|_1,$$

so that each $g \in G$ is an isometry of E_1, and E_1 has enough symmetries. From this it follows that $\sigma(E) \leqq s(E)$.

COROLLARY *If E is real and n-dimensional,* $s(E) \leqq \sqrt{n}$.

For $d(E, l_n^2) \leqq \sqrt{n}$ [11].

The asymmetry constant $s(E)$ introduced here is related to the symmetry constants introduced by Gurarii, Kadec, and Macaev ([8], [9], and [10]). We shall generalize their definitions to apply to finite or infinite-dimensional spaces. Suppose that $B = \{e_i\}$ is a basis for E. If σ is a *finite* permutation of the integers, define the operator $g_\sigma \in L(E,E)$ by $g_\sigma(\Sigma \lambda_i e_i) = \Sigma \lambda_i e_{\sigma(i)}$, and let $G_\sigma(B)$ denote the group of all such operators. If $\varepsilon = (\varepsilon_i)$, where $\varepsilon_i = \pm 1$ and $\varepsilon_i = 1$ for all but finitely many i, define the operator $g_\varepsilon \in L(E,E)$ by $g_\varepsilon(\Sigma \lambda_i e_i) = \Sigma \varepsilon_i \lambda_i e_i$, and let $G_\varepsilon(B)$ denote the group of all such operators. Let $G(B)$ the group generated by $G_\sigma(B)$ and $G_\varepsilon(B)$. The *diagonal asymmetry* $\delta(B)$ of B is defined by

$$\delta(B) = \sup\{\|g\| : g \in G_\sigma(B)\},$$

the *coordinate asymmetry* $\chi(B)$ by $\chi(B) = \sup\{\|g\| : g \in G_\varepsilon(B)\}$, and the *total asymmetry* $\alpha(B)$ by $\alpha(B) = \delta(B)\chi(B)$. Note that, since $G(B)$ is the semi-direct product of $G_\sigma(B)$ and $G_\varepsilon(B)$, $\alpha(B) \geqq \sup\{\|g\| : g \in G(B)\}$. If E is infinite-dimensional, $\delta(B)$ is finite if and only if B is a symmetric basis, in the sense of Singer [17] (cf. [17] Corollary 1 to the main theorem, or [2] Theorem 9), and then $\delta(B) = \sup\{\|g\| : g \in H_\sigma(B)\}$, where H_σ is the group of operators defined by *all* permutations of the integers. A similar argument shows that $\chi(B)$ is finite if and only if B is an unconditional basis, and then $\chi(B) = \sup\{\|g\| : g \in H_\varepsilon(B)\}$, where H_ε is the group of operators defined by *all* sequences $\varepsilon = (\varepsilon_i)$ with $\varepsilon_i = \pm 1$. The *diagonal asymmetry* $\delta(E)$ of E is then defined to be $\inf(\delta(B))$, the infimum being taken over all bases, and the *coordinate asymmetry* $\chi(E)$ and *total asymmetry* $\alpha(E)$ are defined similarly. Note that $\chi(E)$ is the same as the symmetry constant introduced in [13], p. 298. Since $(G(B))' = \{\lambda I_E\}$, it follows that $s(E) \leqq \alpha(E)$. If E is infinite-dimensional, it is not hard to see that $(G_\sigma(B))' = \{\lambda I_E\}$; thus, in this case $s(E) \leqq \delta(E)$.

The following example is of a space E with enough symmetries, but with $\alpha(E) > 1$: Let G_{2n} be the space whose unit ball is the $2n$ sided affine regular polygon in the Minkowsky plane. Representing the vertices by

$$e_i = (\cos(\pi i/n), \sin(\pi i/n)) \quad i = 0,\cdots,2n-1, \text{ then}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos(\pi/n) & -\sin(\pi/n) \\ \sin(\pi/n) & \cos(\pi/n) \end{pmatrix}$$

are matrices which represent isometries of G_{2n}. Obviously, $\{A,B\}' = \{\lambda I_{G_{2n}}\}$.

On the other hand, $\alpha(G_{4n}) = 1$ (e.g. (e_0, e_n) is a symmetric basis), but $\alpha(G_{2(2n+1)}) > 1$.

THEOREM 3. *If E is real and n-dimensional,* $\lambda(E) \leqq 2n\mu(E)\ (s(E))^2$.

This inequality can be obtained by making obvious modifications to the proof of Theorem 1. Alternatively, we can argue as follows. For any $\varepsilon > 0$, there is a linear homeomorphism T of E onto a space F with enough symmetries, such that $\| T \| \ \| T^{-1} \| \leqq s(E) + \varepsilon$. Since $I_E = T^{-1} I_F T$,

$$\lambda(E) = c(I_E) \leqq \| T \| \ \| T^{-1} \| c(I_F) = \| T \| \ \| T^{-1} \| \lambda(F)$$

and $(2\mu(E))^{-1} = \pi_1(E) \leqq \| T \| \ \| T^{-1} \| \pi_1(F) = \| T \| \ \| T^{-1} \| (2\mu(F))^{-1}$. But $\lambda(F) = 2n\mu(F)$, by Theorem 1, and so the result follows by combining the inequalities.

Theorem 3 and the following result enable us to calculate the asymmetry constants of some spaces.

THEOREM 4. *Suppose that $\{X_j\}_{j=1}^n$ is a sequence of Banach spaces. Let $X = \sum_{j=1}^n \oplus X_j$, with the supremum norm, let I_j denote the inclusion mapping: $X_j \to X$, and let P_j denote the projection of X onto X_j, for $j = 1,\cdots,n$.*

(i) *If E is a Banach space and $T\in L(X,E)$, then T is absolutely summing if and only if each map TI_j is; if this is so,*

$$\pi_1(T) = \sum_{j=1}^{n} \pi_1(TI_j).$$

(ii) *If E is a Banach space and $T\in L(E,X)$, then T has the extension property if and only if each map P_jT does; if this is so, $c(T) = \sup_{1\leqq j\leqq n} c(P_jT)$.*

(i) If T is absolutely summing, so is each map TI_j; since $T = \sum_{j=1}^{n} TI_jP_j$, $\pi_1(T) \leqq \sum_{j=1}^{n}\pi_1(TI_jP_j) \leqq \sum_{j=1}^{n}\pi_1(TI_j)$. On the other hand, suppose that each map TI_j is absolutely summing. Then each map TI_jP_j is absolutely summing, and so therefore is $T = \sum_{j=1}^{n} TI_jP_j$. Given $\varepsilon > 0$, for each j there is a finite set S_j in X_j such that $\sup_{\|a\|\leqq 1}\sum_{x\in S_j}|\langle x,a\rangle| = 1$ and $\sum_{x\in S_j}\|TI_jx\| \geqq \pi_1(TI_j)-\varepsilon/n$. Let $S = \bigcup_{j=1}^{n} I_j(S_j)$. Then $\sum_{y\in S}\|Ty\| \geqq \sum_{j=1}^{n}\pi_1(TI_j)-\varepsilon$. On the other hand if $a\in X'$ and $\|a\|\leqq 1$, we can write $a = (a_j)_{j=1}^{n}$, where each $a_j\in X_j'$ and $\sum\|a_j\|\leqq 1$. Thus

$$\sum_{y\in S}|\langle y,a\rangle| = \sum_{j}\sum_{x\in S_j}|\langle x,a_j\rangle| \leqq \sum_{j}\|a_j\| \leqq 1.$$

Hence $\pi_1(T) \geqq \sum_{j=1}^{n}\pi_1(TI_j)-\varepsilon$.

(ii) If T has the extension property, so does each map P_jT, and $c(P_jT)\leqq c(T)$, so that $\sup_{1\leqq j\leqq n} c(P_jT)\leqq c(T)$. On the other hand suppose each P_jT has the extension property. Let L be an isometric embedding of E in a space $C(H)$, where H is compact, Hausdorff and extremally disconnected. Given $\varepsilon>0$, for each j there exists a map S_j from $C(H)$ into X_j such that $\|S_j\|\leqq c(P_jT)+\varepsilon$, and $S_jL = P_jT$. Let $S = \sum_{j=1}^{n} I_jS_j$. Then $\|S\|\leqq \sup c(P_jT)+\varepsilon$, and $SL = \sum I_jP_jT = T$.

COROLLARY. *If $\{X_j\}_{j=1}^{n}$ is a sequence of real finite-dimensional Banach spaces, and if $X = \sum_{j=1}^{n}\oplus X_j$, with the supremum norm, then*

(i) $\mu^{-1}(X) = \sum_{j=1}^{n}\mu^{-1}(X_j)$

and

(ii) $\lambda(X) = \sup_{1\leqq j\leqq n}\lambda(X_j)$.

If we apply the theorem to I_X, we obtain (i) (since clearly $\pi_1(I_j) = \pi_1(I_{X_j})$) and the equation $\lambda(X) = \sup_{1\leqq j\leqq n} c(P_j)$. Since $I_{X_j} = P_jI_j$, $\lambda(X_j)\leqq c(P_j)\|I_j\| = c(P_j)$; since $P_j = I_{X_j}P_j$, $c(P_j)\leqq\lambda(X_j)\|P_j\| = \lambda(X_j)$. Thus we obtain (ii).

REMARK. Using the integral characterization of absolutely p-summing operators, it can also be shown that if $X = \sum_{j=1}^{n} \oplus X_j$, then $(\pi_p(X))^p = \sum_{j=1}^{n}(\pi_p(X_j))^p$. We omit the details.

If f and g are real positive functions defined on the integers, we use the notation $f(n) \lesssim g(n)$ if $\sup_n(f(n)/g(n)) < \infty$, and if in addition $g(n) \lesssim f(n)$, we write $f(n) \sim g(n)$.

For two normed space X and Y, $X \oplus^r Y$ $(1 \leqq r \leqq \infty)$ denotes the space with the norm $\| (x,y) \| = (\| x \|^r + \| y \|^r)^{1/r}$.

THEOREM 5. *If* $1 \leqq p, q, r \leqq \infty$, *then*

$$\alpha(l_n^p \overset{r}{\oplus} l_n^q) \sim s(l_n^p \overset{r}{\oplus} l_n^q) \sim \begin{cases} n^{|1/2p-1/2q|} & ; \text{ if } (p-2)(q-2) \geqq 0 \\ n^{1/2p-1/4} & ; \text{ if } p \leqq 2 \leqq q \text{ and } 1/2p + 1/q \geqq 3/4 \\ n^{1/4-1/2q} & ; \text{ if } p \leqq 2 \leqq q \text{ and } 1/2q + 1/p \leqq 3/4. \end{cases}$$

Let

$$r' = \frac{r}{r-1},\ p' = \frac{p}{p-1},\ q' = \frac{q}{q-1},\ X = l_n^p \overset{\infty}{\oplus} l_n^q,\ Y = l_n^{p'} \overset{\infty}{\oplus} l_n^{q'},$$

and $Z = l_n^p \overset{r}{\oplus} l_n^q$. We need the following facts:

(1) $1 \leqq d(X,Z) \leqq 2$;

(2) $\alpha(E) = \alpha(E^*) \geqq s(E^*) = s(E)$ for a real finite dimensional normed space E;

(3) If E and F are isomorphic Banach spaces, then $\alpha(E) \leqq \alpha(F)d(E,F)$ and $s(E) \leqq s(F)d(E,F)$.

If $2 \leqq p \leqq \infty$ $\lambda(l_n^p) \sim n^{1/p}$ (cf. [16] or [4]), thus, if $2 \leqq p \leqq q \leqq \infty$ we get by Theorem 3 and Corollary to Theorem 4,

$$\begin{aligned}(\alpha(X))^2 &\geqq (s(X))^2 \geqq \lambda(X)\pi_1(X)/2n \sim \lambda(l_n^p)(\pi_1(l_n^p) + \pi_1(l_n^q))/2n \\ &\geqq \lambda(l_n^p)\pi_1(l_n^q)/2n = \lambda(l_n^p)/2\lambda(l_n^q) \sim n^{1/p-1/q}.\end{aligned}$$

On the other hand, let $1/t = 1/2p + 1/2q$, by (1) – (3) and [8] or [4],

$$\begin{aligned}\alpha(X) &\leqq d(X, l_{2n}^t) \leqq d(X, l_n^t \overset{\infty}{\oplus} l_n^t) d(l_n^t \overset{\infty}{\oplus} l_n^t, l_{2n}^t) \\ &\leqq 2d(X, l_n^t \overset{\infty}{\oplus} l_n^t) \leqq 2\max\{d(l_n^t, l_n^p), d(l_n^t, l_n^q)\} \\ &= 2\max\{n^{1/p-1/t}, n^{1/t-1/q}\} = 2n^{1/2p-1/2q}.\end{aligned}$$

Therefore, $\alpha(Z) \sim s(Z) \sim n^{1/2p-1/2q}$.

If $1 \leqq q \leqq p \leqq 2$, we calculate the asymmetry constants of

$$Z^* = l_n^{p'} \overset{r'}{\oplus} l_n^{q'},$$

and since $2 \leqq p' \leqq q' \leqq \infty$, we get by (1) – (3), $\alpha(Z) \sim s(Z) \sim n^{1/2q-1/2p}$.

If $p \leqq 2 \leqq q$, then by results of [4] $\lambda(l_n^p) \sim n^{1/2}$, hence, similarly as above

$$(\alpha(X))^2 \geqq (s(X))^2 \gtrsim \lambda(l_n^p)/\lambda(l_n^q) \sim n^{1/2-1/q},$$

from this and since $q' \leqq 2 \leqq p'$, we obtain by (1)–(3), $s(X) = s(X^*) \gtrsim n^{1/4-1/2p'}$. Combining the inequalities, we have

$$s(Z) \gtrsim \max\{n^{1/2p-1/4}, n^{1/4-1/2q}\} \text{ if } p \leqq 2 \leqq q.$$

If in addition $1/2p + 1/q \geqq 3/4$, let $1/t = 1/2p + 1/4$, then proceeding as above

$$\alpha(X) \leqq d(X, l_{2n}^t) \leqq 2\max\{d(l_n^p, l_n^t), d(l_n^q, l_n^t)\} \sim n^{1/2p-1/4};$$

hence,

$$\alpha(Z) \sim s(Z) \sim n^{1/2p-1/4}.$$

The case $p \leqq 2 \leqq q$, $1/p + 1/2q \leqq 3/4$ is dealt with similarly, where the space considered is Z^*.

REMARK. Theorem 5 solves the problem (cf. [8] p. 722) of finding a sequence of finite dimensional spaces such that $\alpha(X_n)_{n\to\infty} \to \infty$, and suggests the following

CONJECTURE. $s_n = \sup\{s(E); \dim E = n\} = O(n^{1/4})$.

It is easy to see that if we take r to be the integer nearest to $n/3$, then $s(l_r^2 \overset{\infty}{\oplus} l_{n-r}^\infty) \geqq (8n/27\pi)^{1/4}$, hence $cn^{1/4} \leqq s_n \leqq \sqrt{n}$, where $c^4 = 8/27\pi$.

The second named author proves in his Ph.D. Thesis the following related results:

(a) If $1 \leqq p \leqq 2 \leqq q \leqq \infty$ and $n = 2^k$, then

$$s(l_n^p \overset{r}{\oplus} l_n^q) \sim \max\{n^{1/2p-1/4}, n^{1/4-1/2q}\}.$$

(b) $$\alpha(l^p \overset{r}{\oplus} l^q) = \infty \text{ if } p \neq q.$$

The following problems then arise:

PROBLEM 1. *What is the asymptotic behaviour of* $\alpha(l_n^p \overset{r}{\oplus} l_n^q)$ *when* $p \leqq 2 \leqq q$ *and* $1/2p + 1/q < \frac{3}{4} < 1/p + 1/2q$?

PROBLEM 2. *Is* $s(l^p \overset{r}{\oplus} l^q) = \infty$ *when* $p \neq q$?

The existence of a projection of norm 1 of $l^p \overset{r}{\oplus} l^q$ onto $l_n^p \overset{r}{\oplus} l_n^q$ and the fact that $s(l_n^p \overset{r}{\oplus} l_n^q) \to_{n\to\infty} \infty$ if $p \neq q$, need not imply that $s(l^p \overset{r}{\oplus} l^q) = \infty$. We are grateful to Dr. M. A. Perles for noting this and supplying an example of a space with enough symmetries which admits a projection of norm 1 onto a subspace with a large asymmetry constant:

Let $E = \sum_{j=1}^{m} \overset{\infty}{\oplus} X_j$, where $X_j = X$ $(j = 1,2,\cdots,m)$ is a space with enough symmetries. Then E is a space with enough symmetries.

For, let $P_j : E \to X_j$ and $I_j : X_j \to E$ be the projection and inclusion operators. For any permutation $\sigma = (\sigma(1),\cdots,\sigma(m))$, let $g_\sigma = \sum_{j=1}^{m} I_{\sigma(j)}P_j$, and for any $\varepsilon = (\varepsilon_1,\cdots,\varepsilon_m)$ where $\varepsilon_i = \pm 1$, let $g_\varepsilon = \sum_{j=1}^{m} \varepsilon_j I_j P_j$. If g is an isometry of X, let $\tilde{g} = \sum_{j=1}^{m} I_j g P_j$.

Then the group G generated by all g_σ, g_ε and $\tilde{g}$ is a group of isometries such that $G' = \{\lambda I_E\}$.

Take now $X = l_n^2$, $m = n + 1$. There is a natural projection of norm 1 of E onto $l_n^2 \overset{\infty}{\oplus} l_n^\infty$, and $s(l_n^2 \overset{\infty}{\oplus} l_n^\infty) \to \infty$.

4. Absolutely p-summing and p-nuclear constants

Let us now apply the duality results of [14] concerning p-nuclear and absolutely p-summing operators. If $1 \leqq p < \infty$, we recall that $\pi_p(E,F)$ denotes the set of all $T \in L(E,F)$ for which there is a constant C such that

$$\left(\sum_{i=1}^{m} \| Te_i \|^p\right)^{1/p} \leqq C \sup\left\{\left(\sum_{i=1}^{m} |\langle e_i, e'\rangle|^p\right)^{1/p} : \|e'\| \leqq 1\right\}$$

for each finite sequence $(e_1,\cdots,e_m)$ in E. $\pi_p(E,F)$ is a Banach space under the norm $\pi_p(T) = \inf C$([14], [15], [13]); if $T \in \pi_p(E,F)$, T is said to be *absolutely p-summing*.

If $1 < p < \infty$, $N_p(E,F)$ denotes the set of all $T \in L(E,F)$ which can be written in the form $T(e) = \sum_{i=1}^{\infty} \langle e, e_i'\rangle f_i$, where $(\sum_{i=1}^{\infty} \| e_i' \|^p)^{1/p} < \infty$ and

$$\sup\left\{\left(\sum_{i=1}^{\infty} |\langle f_i, f'\rangle|^{p'}\right)^{1/p'} : \|f'\| \leqq 1\right\} < \infty$$

(where $1/p + 1/p' = 1$). $N_p(E,F)$ is a Banach space under the norm

$$\nu_p(T) = \inf\left\{\left(\sum_{i=1}^{\infty} \| e_i' \|^p\right)^{1/p} \sup\left\{\left(\sum_{i=1}^{\infty} |\langle f_i, f'\rangle|^{p'}\right)^{1/p'} : \|f'\| \leqq 1\right\}\right\},$$

where the infimum is taken over all representations of T [14]; if $T \in N_p(E,F)$, T is said to be *p-nuclear*. Finally, we recall that $T \in L(E,F)$ is *nuclear* if it can be written in the form $T(e) = \sum_{i=1}^{\infty} \langle e, e_i'\rangle f_i$, where $\sum_{i=1}^{\infty} \| e_i' \| \, \| f_i \| < \infty$; the collection $N_1(E,F)$ of all nuclear operators from E to F is a Banach space under the norm $\nu_1(T) = \inf(\sum_{i=1}^{\infty} \| e_i' \| \, \| f_i \|)$, where again the infimum is taken over all representations of T.

If E is finite dimensional, we write $\pi_p(E)$ for $\pi_p(I_E)$, $\nu_p(E)$ for $\nu_p(I_E)$.

We shall apply the following results of [14] to the case where $E = F$ and $\dim E = n < \infty$:

(α) If $1 < p < \infty$, $\pi_p(T) \leqq \nu_p(T)$ ([14] Satz 35 and Satz 40);

(β) $\pi_2(T) = \nu_2(T)$ ([14] Satz 35, and the remark on p. 43);

(γ) If $1 \leqq p \leqq q < \infty$, $\pi_p(T) \geqq \pi_q(T)$ and $\nu_p(T) \geqq \nu_q(T)$ ([14] Satz 31 and Satz 4);

(δ) If $1 < p < \infty$, and $1/p + 1/p' = 1$, then $\nu_p(T) = \sup\{Tr(ST)\,;\ \pi_{p'}(S) \leqq 1\}$ ([14] Satz 52) and $\pi_p(T) = \sup\{Tr(ST);\ \nu_{p'}(S) \leqq 1\}$.

We shall also need

(ε) $\nu_1(T) = \sup\{Tr(ST);\ \|S\| \leqq 1\}$ ([6] Théorème 1).

THEOREM 6. *If $1 < p < \infty$, $1/p + 1/p' = 1$ and $n = \dim E$, then*

$$n \leqq \pi_p(E)\nu_{p'}(E) \leqq n(s(E))^2.$$

This is proved in exactly the same way as Theorem 3.

COROLLARY. *If E has enough symmetries, then*

$$n = \pi_p(E)\nu_{p'}(E).$$

We shall also use the following result due to John [11]:

THEOREM (JOHN). *Let $(F, \|\ \|_F)$ be a real n-dimensional Banach space with unit ball S_F. Let $\|\ \|_2$ be the Hilbert space norm on F with the property tha the unit ball in $(F, \|\ \|_2)$ is the ellipsoid of least volume containing S_F. Then there exists an integer s, vectors $y_1, \cdots, y_s$ in F and positive scalars $\lambda_1, \cdots, \lambda_s$ such that*

(i) $n \leqq s \leqq n(n+1)/2$,

(ii) $1 = \|y_r\|_F = \|y_r\|_2$ *for* $1 \leqq r \leqq s$,

and

(iii) $\sum_{r=1}^{s} \lambda_r \langle x, y_r \rangle y_r = x$ *for all x in F*

($\langle x, y_r \rangle$ is the inner product defined by $\|\ \|_2$).

Note that it follows from (ii) and (iii) that $\sum_{r=1}^{s} \lambda_r = n$ and that $\|y_r\|_{F^*} = 1$, where $\|\ \|_{F^*}$ is the dual norm of the dual space F^*.

THEOREM 7. *If E is an n-dimensional real Banach space, then*

(i) $\pi_2(E) = \sqrt{n}$;

(ii) $v_1(E) = n$;

(iii) $n^{1/2} \leqq \pi_p(E) \leqq v_p(E) \leqq n^{1/p}$ *for* $1 \leqq p \leqq 2$;

(iv) $n^{1/q} \leqq \pi_q(E) \leqq v_q(E) \leqq n^{1/2}$ *for* $2 \leqq q < \infty$;

(v) $\lambda(E) \leqq n^{1/2}$.

Applying (β) to Theorem 6, it follows that $\pi_2(E) \geqq \sqrt{n}$, and it follows directly from (ε) that $v_1(E) \geqq n$. Now let $F = E^*$, and let y_r, λ_r be the quantities in John's Theorem. Then if $1 \leqq p \leqq 2$,

$$x = \sum_{r=1}^{s} \langle x, \lambda_r^{1/p} y_r \rangle \lambda_r^{1/p'} y_r, \text{ for any } x \text{ in } E,$$

(where $1/p + 1/p' = 1$), so that

$$\begin{aligned} v_p(E) &\leqq \left(\sum_{r=1}^{s} \| \lambda_r^{1/p} y_r \|_F^p\right)^{1/p} \sup_{\|z\|_F \leqq 1} \left(\sum_r |\langle \lambda_r^{1/p'} y_r, z\rangle|^{p'}\right)^{1/p'} \\ &= \left(\sum_{r=1}^{s} \lambda_r\right)^{1/p} \sup_{\|z\|_F \leqq 1} \left(\sum_r \lambda_r |\langle y_r, z\rangle|^{p'}\right)^{1/p'} \\ &\leqq n^{1/p} \sup_{\|z\|_F \leqq 1} \left(\sum_r \lambda_r \langle y_r, z\rangle^2\right)^{1/p'} \\ &= n^{1/p} \sup_{\|z\|_F \leqq 1} \| z \|_2^{2/p'} = n^{1/p}, \end{aligned}$$

since $|\langle y_r, z\rangle| \leqq 1$ and $p' \geqq 2$. This proves (i) and (ii), and since $\pi_p(E) \geqq \pi_2(E)$ (by (γ)), (iii) is also established. If $q \geqq 2$, $(\pi_q(E))^q \geqq (\pi_2(E))^2 = n$ by [5] Theorem 1, Corollary 1, and $v_q(E) \leqq v_2(E) = n^{1/2}$, and so we obtain (iv). Finally,

$$\begin{aligned} \lambda(E) &= \sup\{ |Tr(T)| : T \in L(E,E), \pi_1(T) \leqq 1\} \\ &\leqq \sup\{ |Tr(T)| : T \in L(E,E), \pi_2(T) \leqq 1\} \\ &= \pi_2(E), \end{aligned}$$

and (v) follows from this.

Remarks. The validity of the estimate $\lambda(E) \leqq n^{1/2}$ was first noticed by Kadec [20] who also used John's theorem. We are indebted to A. Pełczyński for this information. This observation of Kadec drew our attention to John's theorem and suggested its application in Theorem 7.

The inequality $\pi_1(E) \geqq n^{1/2}$ can be considered as the solution to a weak form of the conjecture due to Dvoretzky and Rogers ([1], Remark p. 196). Theorem 7 also improves the estimates of [5] Theorem 3 ((6) and (7)) and of [8] Corollary 3, Theorem 6 and Corollary 4 ([10], Theorem 3 and its corollary). Note also that these inequalities are asymptotically exact (cf. [5] Theorem 2, (2) and (3)).

Another conclusion which can be drawn from John's Theorem is :

THEOREM 8. *Let X be a closed subspace of a Banach space Y, such that $\dim(Y/X) = n$. For any $\varepsilon > 0$, there is a projection P of Y onto X such that $\| P \| \leqq 1 + \sqrt{n} + \varepsilon$.*

Let $Z = Y/X$. We formulate John's Theorem in the following way : There is an isomorphism $T: l_n^2 \overset{\text{onto}}{\to} Z$ such that $\| T^{-1} \| = 1$, and there are s points $e_1, \cdots, e_s$ in l_n^2, and positive scalars $\lambda_1, \cdots, \lambda_s$ satisfying the following equations

(i) $\sum_r \lambda_r = n$,

(ii) $1 = \| e_r \| = \| Te_r \| = \| T^{*-1} e_r \|$ for $1 \leqq r \leqq s$,

(iii) $\sum_{r=1}^{s} \lambda_r \langle x, e_r \rangle e_r = x$ for all x in l_n^2.

Let $U : Y \to Z$ be the natural mapping $Uy = y + X$, and let $z_r = Te_r$. There exist y_r in $U^{-1}Z$ such that $Uy_r = z_r$ and $\| y_r \| \leqq 1 + \varepsilon$. Define the operator $P : Y \to Y$ by

$$Py = y - \Sigma \lambda_r \langle Uy, T^{*-1} e_r \rangle y_r .$$

We shall see that P is a projection of Y onto X. Indeed, $Tx = \Sigma \lambda_r \langle Tx, T^{*-1} e_r \rangle Te_r$ for all x in l_n^2, so that $Uy = \Sigma \lambda_r \langle Uy, T^{*-1} e_r \rangle Uy_r$ for all $y \in Y$. This implies that $UPy = 0$; therefore P maps Y into X. But since $Ux = 0$, if $x \in X$, it follows that $Px = x$, whence P is a projection of Y onto X.

Suppose now that $y \in Y$ and $\| y \| \leqq 1$, then $\| Uy \| \leqq 1$, and since $\| y_r \| \leqq 1 + \varepsilon$ we get

$$\| Py \| \leqq 1 + (1 + \varepsilon) \Sigma \lambda_r | \langle Uy, T^{*-1} e_r \rangle | \leqq 1 + (1 + \varepsilon)(\Sigma \lambda_r)^{1/2} (\Sigma \lambda_r \langle Uy, T^{*-1} e_r \rangle^2)^{1/2}.$$

Now, $Uy = \Sigma \lambda_r \langle Uy, T^{*-1} e_r \rangle Te_r$; therefore, $T^{*-1} T^{-1} Uy = \Sigma \lambda_r \langle Uy, T^{*-1} e_r \rangle T^{*-1} e_r$; hence, $(\Sigma \lambda_r \langle Uy, T^{*-1} e_r \rangle^2)^{1/2} = \langle Uy, T^{*-1} T^{-1} Uy \rangle^{1/2} = \| T^{-1} Uy \| \leqq 1$.

From this it follows that $\| P \| \leqq 1 + (1 + \varepsilon) \sqrt{n}$, and the proof is concluded.

Theorem 8 improves on the result of [19], Theorem 6. (There, the estimate $\|P\| \leqq 1 + n + \varepsilon$ is obtained). An independent proof of Theorem 8 was obtained by Davies [21].

THEOREM 9. *If E is an n-dimensional subspace of l^p $(1 \leqq p \leqq 2)$, then $\lambda(E) \geqq K_G^{-1}\sqrt{n}$ (where K_G is the Grothendieck universal constant [6] [13]).*

Let $J: E \to C(X)$ be an embedding operator, where X is a suitable compact Hausdorff space, and let P be a projection of $C(X)$ onto $J(E)$. Applying (β) to (δ), with $T = S = P$, we have $(\pi_2(P))^2 \geqq Tr(P^2) = Tr(P) = n$. Since $J^{-1}P$ maps $C(X)$ into l^p and since $C(X)$ is an $\mathscr{L}_{\infty,1}$ space, we obtain from [13], Theorem 4.3:

$$\pi_2(P) = \pi_2(J^{-1}P) \leqq K_G\|J^{-1}P\| = K_G\|P\|.$$

COROLLARY. *If E is n-dimensional,*

$$\lambda(E)\lambda(E^*) \geqq K_G^{-1}\sqrt{n}.$$

Without loss of generality we may assume that E is a polyhedral space; let $J: E \to l_m^\infty$ be any suitable embedding operator. Let $P: l_m^\infty \to JE$ be a projection such that $\|P\| = \lambda(E)$. P^*J^* is a projection of $(l_m^\infty)^* = l_m^1$ onto P^*E^*, and by Theorem 9 $\lambda(P^*E^*) \geqq K_G^{-1}\sqrt{n}$; since $\lambda(P^*E^*) \leqq \lambda(E^*)d(E^*, P^*(E^*)) \leqq \lambda(E^*)\|P^*\| = \lambda(E^*)\lambda(E)$, the result follows.

5. The constants associated with l_n^r spaces

A detailed account of the absolutely p-summing and projection constants of the spaces l_n^r was given in [4] and [5]. We now show how these results, together with results concerning the p-nuclear constants, can be obtained in a slightly stronger form, using the duality described in the preceding section. If $1 < s < \infty$, we denote by s' the number $s/(s-1)$.

THEOREM 10. *There exists a function $B(q) = O(q^{1/2})$ such that*

(i) $\nu_1(l_n^r) = n$ *for* $1 \leqq r \leqq \infty$;

(ii) $\sqrt{n} \geqq \lambda(l_n^r) \geqq K_G^{-1}\sqrt{n}$, *for* $1 \leqq r \leqq 2$;

(iii) $K_G\sqrt{n} \geqq \pi_p(l_n^r) \geqq \pi_2(l_n^r) = \sqrt{n} \geqq \pi_q(l_n^r) \geqq B(q)^{-1}\sqrt{n}$,
for $1 \leqq r \leqq 2$, $1 \leqq p \leqq 2 \leqq q < \infty$;

(iv) $B(p')\sqrt{n} \geqq \nu_p(l_n^r) \geqq \nu_2(l_n^r) = \sqrt{n} \geqq \nu_q(l_n^r) \geqq K_G^{-1}\sqrt{n}$,
for $1 \leqq r \leqq 2$, $1 < p \leqq 2 < q < \infty$;

(v) $\lambda(l_n^r) = 2\alpha_r(n)n^{1/r}$ *for* $r \geqq 2$;

(vi) $n^{1/r} \geqq \pi_p(l_n^r) \geqq (B(p))^{-1} n^{1/r}$ *for* $2 \leqq r \leqq p < \infty$;

(vii) $n^{1/r} \geqq \nu_p(l_n^r) \geqq 2\alpha_r(n) n^{1/r}$ *for* $2 \leqq r \leqq p < \infty$;

(viii) $n^{1/p} = \nu_p(l_n^r) = \pi_p(l_n^r)$ *for* $r' \leqq p \leqq r \leqq \infty$;

(ix) $(2\alpha_r(n))^{-1} n^{1/r'} = \pi_1(l_n^r) \geqq \pi_p(l_n^r) \geqq n^{1/r'}$ *for* $1 \leqq p \leqq r' \leqq r \leqq \infty$;

(x) $B(p') n^{1/r'} \geqq \nu_p(l_n^r) \geqq n^{1/r'}$ *for* $1 \leqq p \leqq r' \leqq r < \infty$;

(i) and (ii) follow from Theorems 7, and 9, and so do all the results of (iv), except for the first inequality.

Let $E = \{\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) : \varepsilon_i = \pm 1, \text{ for } 1 \leqq i \leqq n\}$. If $x \in l_n^r$, then

$$x = 2^{-n} \sum_{\varepsilon \in E} \langle x, \varepsilon \rangle \varepsilon,$$

so that

$$\begin{aligned} \nu_p(l_n^r) &\leqq 2^{-n} \Big(\sum_{\varepsilon \in E} \| \varepsilon \|_{r}^{p} \Big)^{1/p} \sup \left\{ \Big(\sum_{\varepsilon \in E} | \langle \varepsilon, f' \rangle |^{p'} \Big)^{1/p'} : \| f' \|_{r'} \leqq 1 \right\} \\ &= 2^{-n/p'} n^{1/r} \sup \left\{ \Big(\sum_{\varepsilon \in E} | \langle \varepsilon, f' \rangle |^{p'} \Big)^{1/p'} : \| f' \|_{r'} \leqq 1 \right\}. \end{aligned}$$

Now,

$$\Big(\sum_{\varepsilon \in E} | \langle \varepsilon, f' \rangle |^{p'} \Big)^{1/p'} = 2^{n/p'} \left(\int_0^1 \Big| \sum_{i=1}^n f_i \phi_{i-1}(t) \Big|^{p'} dt \right)^{1/p'}$$

where ϕ_i is the i-th Rademacher function, and by [18] Chapter V Theorem 8.4, there exists a function $B(p) = O(p^{1/2})$ such that

$$\left(\int_0^1 \Big| \sum_{i=1}^n f_i \phi_{i-1}(t) \Big|^{p'} dt \right)^{1/p'} \leqq B(p') \| f' \|_2 .$$

Now, if $1 \leqq r \leqq 2$, $\| f' \|_2 \leqq n^{1/2 - 1/r'} \| f' \|_{r'}$, so that combining the inequalities, $\nu_p(l_n^r) \leqq B(p') \sqrt{n}$. Thus (iv) is established, and (iii) follows by duality. If $r \geqq 2$, $\| f' \|_2 \leqq \| f' \|_{r'}$, so that $\nu_p(l_n^r) \leqq B(p') n^{1/r'}$, giving the first inequality in (x); the first inequality in (vi) is due to [5] (Equation (24)), and (vi) and (x) are now deduced by duality. The last inequality in (ix) is established in [5] (Equation 26), and the expression for $\pi_1(l_n^r)$ is due to Rutovitz [16]; (v) was deduced from this in §2 and (vii) follows from (ix) by duality.

Finally, we can write $I(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i$, so that if $p \geqq r$,

$$\begin{aligned} \nu_p(l_n^r) &\leqq \Big(\sum_{i=1}^n \| e_i \|_{r}^{p} \Big)^{1/p} \sup \left\{ \Big(\sum_{i=1}^n | \langle e_i, f' \rangle |^{p'} \Big)^{1/p'} : \| f' \|_{r'} \leqq 1 \right\} \\ &= n^{1/p} \sup \{ \| f' \|_{p'} : \| f' \|_{r'} \leqq 1 \} = n^{1/p}; \end{aligned}$$

(viii) follows from this, from Theorem 7 and Corollary to Theorem 6.

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St. John's College, Cambridge, and
Lehigh University, Bethlehem, Pennsylvania
The Hebrew University of Jerusalem